Name: Date:

# 7C: Partial Fractions

**<u>Consider the integral</u>** $\int \frac{1}{x^2+5x+6}dx$

We can quickly see that u-substitution doesn't work, and trigonometric substitution will get quite messy. However, notice that $x^2+5x+6=(x+2)(x+3)$, and furthermore

$$\int \frac{1}{x^2+5x+6}dx = \int \frac{1}{(x+2)(x+3)}dx = \int \left(\frac{1}{x+2}-\frac{1}{x+3}\right)dx$$

By breaking the denominator up into factors, we get a sum that we can evaluate:

$$\int \left(\frac{1}{x+2}-\frac{1}{x+3}\right)dx = \int \frac{1}{x+2}dx - \int \frac{1}{x+3}dx = \ln|x+2| - \ln|x+3| + c$$

We call this technique of breaking a fraction up into a sum of fractions, ***"Partial Fraction Decomposition"***, and the ***partial fractions*** in this example are $\frac{1}{x+2}$ and $\frac{1}{x+3}$.

So, how do we find these two partial fractions? Let's try another integral.

**<u>Example:</u> Evaluate the integral**

$$\int \frac{x+20}{x^2+11x+18}dx$$

- Step 1: Factor Denominator: $\frac{x+20}{x^2+11x+18} = \frac{x+20}{(x+2)(x+11)}$
- Step 2: Set up Partial Fractions with generic Denominators $\frac{x+20}{(x+2)(x+11)} = \frac{A}{x+2} + \frac{B}{x+11}$
- Step 3: Work "Backwards" to solve for numerators

$$\frac{x+20}{(x+2)(x+11)} = \frac{A}{x+2} + \frac{B}{x+11} = \frac{A(x+11)}{(x+2)(x+11)} + \frac{B(x+2)}{(x+11)(x+2)} = \frac{A(x+11)+B(x+2)}{(x+2)(x+11)}$$

so…

$$x+20 = A(x+11) + B(x+2)$$
$$x+20 = Ax + 11A + Bx + 2B$$
$$1x + 20 = (A+B)x + (11A+2B)$$
$$1 = A+B \ , and \ \ 20 = 11A+2B$$

Solve this system of equations to get

$$\begin{cases} 2 = 2A+2B \\ 20 = 11A+2B \end{cases} \rightarrow 18 = 9A \ \rightarrow 2 = A, -1 = B$$

- Step 4: Rewrite integral and evaluate

$$\int \frac{x+20}{x^2+11x+18}dx = \int \frac{2}{x+2} - \frac{1}{x+11}dx = 2\ln|x+2| - \ln|x+11| + C$$

Example 2: Multiple Linear Factors. Evaluate

$$\int \frac{5x^2 + 20x + 6}{x^3 + 2x^2 + x} dx$$

Example 3: Linear and Quadratic Factors. Evaluate

$$\int \frac{2x^3 - 4x - 8}{(x^2 - x)(x^2 + 4)} dx$$

**Summary of Key Techniques:**

**DECOMPOSITION OF $N(x)/D(x)$ INTO PARTIAL FRACTIONS**

1. **Divide if improper:** If $N(x)/D(x)$ is an improper fraction (that is, if the degree of the numerator is greater than or equal to the degree of the denominator), divide the denominator into the numerator to obtain

$$\frac{N(x)}{D(x)} = (\text{a polynomial}) + \frac{N_1(x)}{D(x)}$$

where the degree of $N_1(x)$ is less than the degree of $D(x)$. Then apply Steps 2, 3, and 4 to the proper rational expression $N_1(x)/D(x)$.

2. **Factor denominator:** Completely factor the denominator into factors of the form

$$(px + q)^m \qquad \text{and} \qquad (ax^2 + bx + c)^n$$

where $ax^2 + bx + c$ is irreducible.

3. **Linear factors:** For each factor of the form $(px + q)^m$, the partial fraction decomposition must include the following sum of $m$ fractions.

$$\frac{A_1}{(px + q)} + \frac{A_2}{(px + q)^2} + \cdots + \frac{A_m}{(px + q)^m}$$

4. **Quadratic factors:** For each factor of the form $(ax^2 + bx + c)^n$, the partial fraction decomposition must include the following sum of $n$ fractions.

$$\frac{B_1x + C_1}{ax^2 + bx + c} + \frac{B_2x + C_2}{(ax^2 + bx + c)^2} + \cdots + \frac{B_nx + C_n}{(ax^2 + bx + c)^n}$$

**Here are some steps** for solving the basic equation found by the partial fraction decomposition.

**GUIDELINES FOR SOLVING THE BASIC EQUATION**

*Linear Factors*

1. Substitute the roots of the distinct linear factors in the basic equation.
2. For repeated linear factors, use the coefficients determined in guideline 1 to rewrite the basic equation. Then substitute other convenient values of $x$ and solve for the remaining coefficients.

*Quadratic Factors*

1. Expand the basic equation.
2. Collect terms according to powers of $x$.
3. Equate the coefficients of like powers to obtain a system of linear equations involving $A$, $B$, $C$, and so on.
4. Solve the system of linear equations.

Example 4: Repeated Linear Factors. Evaluate

$$\int \frac{5x^2 + 20x + 6}{x^3 + 2x + x}\, dx$$